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A NOTE ON BOUNDARY CONDITIONS FOR USE WITH THE DIFFERENTIAL APPROXIMATION TO RADIATIVE TRANSFER*

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NOMENCLATURE

$B(T)$,	black body steradiancy, $\sigma T^4/\pi$;
$F_j(\eta)$,	defined by equation (14);
$G(\eta, \eta')$,	defined by equation (13);
I,	frequency integrated specific intensity;
I_n,	nth angular moment of the specific intensity;
J,	frequency integrated source function;
n,	outward normal to a surface;
q,	heat flux;
s,	distance along a surface;
T,	temperature;
$\alpha_a(\vec{\Omega}, \vec{r}, t)$,	mean absorption coefficient, equation (4);
$\alpha_e(\vec{r}, t)$,	mean emission coefficient, equation (3);
$\alpha_a^{(n)}(\vec{r}, t)$,	nth tensor moment of α_a, equation (5);
ϵ,	emissivity;
η,	optical depth, equation (10);
κ,	temperature slip coefficient, equation (29);
τ_l,	Bouguer number, $\alpha_\infty l$;
ϕ,	heat flux potential, equation (7).

FORMULATION OF CONSISTENT ONE-DIMENSIONAL BOUNDARY CONDITIONS

THE LOWEST order full-range differential formulation of RGD which yields both the correct optically thick and emission dominated limits is governed by the following equations if the speed of light is much larger than any velocity scale of interest [1–3].

$$\operatorname{div}(\vec{q}) = \tau_l\{4\pi\, \mathrm{d}_e J - \alpha_a^{(0)} I_0\} \tag{1}$$

$$\operatorname{grad}(I_0) = -3\tau_l \alpha_a^{(1)} \vec{q} \tag{2}$$

where the heat flux, $\vec{q}$, first moment of the specific intensity, I_0, and integrated source function, J, are normalized by the reference emission level, σT_∞^4. All lengths are normalized by the dimension of the physical region of interest, l, and absorption coefficients by an absorption level, α_∞. The emission coefficient,

$$\alpha_e = \int^{\infty} \alpha_\nu J_\nu\, \mathrm{d}\nu / J, \qquad J = \int^{\infty} J_\nu\, \mathrm{d}\nu \tag{3}$$

is merely the Planck mean if the medium is in local thermodynamic equilibrium (LTE).

The absorption coefficients $\alpha_a^{(0)}$ and $\alpha_a^{(1)}$ are the first two tensor moments of the general absorption coefficient $\alpha_a(\vec{\Omega}, \vec{r}, t)$ and are discussed elsewhere [3]. If $\vec{\Omega}$ is the unit vector in the direction of propagation and Ω^n is interpreted as a general tensor of order n, then if

$$\alpha_a(\vec{\Omega}, \vec{r}, t) = \int^{\infty} \alpha_\nu I_\nu(\vec{\Omega}, \vec{r}, t)\, \mathrm{d}\nu / I, \qquad I = \int^{\infty} I_\nu\, \mathrm{d}\nu \tag{4}$$

where I_ν is the spectral specific intensity, it follows that

$$\alpha_a^{(n)}(r, t)\, \vec{I}_n(\vec{r}, t) = \int_{4\pi} \alpha_a(\vec{\Omega}, \vec{r}, t)\, \vec{\Omega}^n I(\vec{\Omega}, \vec{r}, t)\, \mathrm{d}\vec{\Omega} \tag{5}$$

where

$$\vec{I}_n(\vec{r}, t) = \int_{4\pi} \vec{\Omega}^n I(\vec{\Omega}, \vec{r}, t)\, \mathrm{d}\vec{\Omega}. \tag{6}$$

The consistent inclusion of $\alpha_a^{(0)}$ and $\alpha_a^{(1)}$ has the same effect upon the governing equations that anisotropic scattering would.

Mark overcame the ambiguity in the choice of boundary conditions for black surfaces in one-dimensional situations by considering the surface to emit as a gas and matching moments at the interface [4]. The procedure may be generalized as follows. Consider first an infinite expanse in which there are initially no material boundaries, but in which the temperature and pressure fields may be nonuniform. A heat flux potential similar to that used by Cohen [5] and Traugott [6] may be defined

$$\vec{q} = \frac{1}{\alpha_a^{(1)}} \operatorname{grad} \phi. \tag{7}$$

Equation (2) requires that

$$I_0 = -3\tau_l \phi \tag{8}$$

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so that equation (1) becomes

$$\operatorname{div}\left(\frac{1}{\alpha_a^{(1)}}\operatorname{grad}\phi\right) - 3\tau_l^2\alpha_a^{(0)}\phi = 4\pi\tau_l\alpha_e J. \tag{9}$$

In a one-dimensional situation the appropriate optical variable is

$$\eta = \int_{R(t)}^{r} \sqrt{[\alpha_a^{(0)}(r,t)\,\alpha_a^{(1)}(r,t)]}\,\mathrm{d}r' \tag{10}$$

where the parameter $R(t)$ is introduced for reasons which will soon be apparent. Equation (9) may be written in the form

$$\frac{\partial^2\phi}{\partial\bar{\eta}^2} - \phi = \frac{4\pi}{3\tau_l}\frac{\alpha_e}{\alpha_a^{(0)}}J - \frac{q}{\tau_l\sqrt{3}}\sqrt{\left(\frac{\alpha_a^{(1)}}{\alpha_a^{(0)}}\right)}\frac{\partial}{\partial\eta}\left\{\log\left[r^j\sqrt{\frac{\alpha_a^{(0)}}{\alpha_a^{(1)}}}\right]\right\} \tag{11}$$

where $\bar{\eta} \equiv (\sqrt{3})\cdot\tau_l\eta$ and $j = 0, 1,$ or 2 for planar, cylindrical, or spherically symmetric cases respectively.

Symmetry requires that $q \to 0$ as $r \to 0$. Since the flux must also vanish far from the source of the disturbance, equation (11) may be transformed into the following integral equation.

$$\phi(\bar{\eta}) = -\frac{1}{3\tau_l}\int_{\bar{\eta}_w}^{\infty} G(\bar{\eta},\bar{\eta}')\,F_j(\bar{\eta}')\,\mathrm{d}\bar{\eta}' \tag{12}$$

where $\bar{\eta}_w = -\bar{\eta}(r = 0)$ and,

$$G(\bar{\eta},\bar{\eta}') = \begin{cases}\cosh(\bar{\eta}' + \bar{\eta}_w)\exp[-(\bar{\eta} + \bar{\eta}_w)] & \bar{\eta} > \bar{\eta}' \\ \cosh(\bar{\eta} + \bar{\eta}_w)\exp[-(\bar{\eta}' + \bar{\eta}_w)] & \bar{\eta} < \bar{\eta}'\end{cases} \tag{13}$$

$$F_j(\bar{\eta}) = 4\pi\frac{\alpha_e}{\alpha_a^{(0)}}J - \sqrt{\left(\frac{3\alpha_a^{(1)}}{\alpha_a^{(0)}}\right)}\,q\,\frac{\partial}{\partial\bar{\eta}}\left\{\log\left[r^j\sqrt{\left(\frac{\alpha_a^{(0)}}{\alpha_a^{(1)}}\right)}\right]\right\}. \tag{14}$$

Suppose that the gas contained within $0 \leqslant r \leqslant R$ (or an isolated slab $-R \leqslant r \leqslant R$ in the planar case) is maintained isothermal, that it is separated from its surroundings by a fictitious membrane, and that the absorption properties therein and temperature, T_w, are at one's disposal. If all incident radiation is to be absorbed by the isolated gas, $\alpha_e \to \alpha_a^{(0)} \to \alpha_a^{(1)} \to \infty$ therein, and

$$I_0(\eta) = 2\pi B(T_w)\exp[-(\sqrt{3})\tau_l\eta] + \frac{\tau_l\sqrt{3}}{2}\int_0^{\infty}\exp[-(\sqrt{3})\tau_l|\eta - \eta'|]\,F_j(\eta')\,\mathrm{d}\eta' \tag{15}$$

$$\sqrt{\left(\frac{\alpha_a^{(1)}}{\alpha_a^{(0)}}\right)}q(\eta) = \frac{2\pi}{\sqrt{3}}B(T_w)\exp[-(\sqrt{3})T_l\eta] + \frac{\tau_l}{2}\int_0^{\infty}\operatorname{sgn}(\eta - \eta')\exp[-(\sqrt{3})\tau_l|\eta - \eta'|]\,F_j(\eta')\,\mathrm{d}\eta' \tag{16}$$

It follows that at the surface, $r = R(t)$

$$I_{0_w} + \sqrt{\left(\frac{3\alpha_1^{(1)}}{\alpha_a^{(0)}}\right)}q_w = 4\pi B(T_w), \qquad B = T^4/\pi \tag{17}$$

which is Mark's boundary condition for a black wall.

A diffusely reflecting surface is easily simulated either by assuming the isolated volume to be void or imposing the condition that there can be no flux into the surface. The result is

$$I_0(\eta) = \tau_l(\sqrt{3})\left\{\int_0^{\eta}\cosh[(\sqrt{3})\tau_l\eta']\exp[-(\sqrt{3})\tau_l\eta']\,F_j(\eta')\,\mathrm{d}\eta' + \int_{\eta}^{\infty}\cosh[(\sqrt{3})\tau_l\eta]\exp[-(\sqrt{3})\tau_l\eta']\,F_j(\eta')\,\mathrm{d}\eta'\right\} \tag{18}$$

$$\sqrt{\left(\frac{\alpha_a^{(1)}}{\alpha_a^{(0)}}\right)}q(\eta) = \tau_l\left\{\int_0^{\eta}\cosh[(\sqrt{3})\tau_l\eta']\exp[-(\sqrt{3})\tau_l\eta] \times F_j(\eta')\,\mathrm{d}\eta' - \int_{\eta}^{\infty}\sinh[(\sqrt{3})\tau_l\eta]\exp[-(\sqrt{3})\tau_l\eta'] \times F_j(\eta')\,\mathrm{d}\eta'\right\} \tag{19}$$

Analogous to energy accommodation in kinetic theory, it is postulated that the linearity of equation (1) and (2) in I_0 and q may be used to state that

$$\begin{Bmatrix}I_0\\q\end{Bmatrix}_w = \epsilon\begin{Bmatrix}I_0\\q\end{Bmatrix}_{w_{\text{black}}} + (1 - \epsilon)\begin{Bmatrix}I_0\\q\end{Bmatrix}_{w_{\text{diffuse}}} \tag{20}$$

where e is the emissivity of the surface. It will be shown that the analogy is precise, and the result is

$$I_{0_w} + \left(\frac{2-\epsilon}{\epsilon}\right)\sqrt{\left(\frac{3\alpha_a^{(1)}}{\alpha_a^{(0)}}\right)}q_w = 4\pi B(T_w) \tag{21}$$

$$I_{0_w} = 2\pi\epsilon B(T_w) + \left(1 - \frac{\epsilon}{2}\right)\tau_l(\sqrt{3})\int_0^{\infty}\exp[-(\sqrt{3})\tau_l\eta']\,F_j|\eta')\,\mathrm{d}\eta'. \tag{22}$$

Equation (21) may be derived if one considers the half-range intensities to be discrete streams

$$I^{(\pm)} = I_0 \pm \sqrt{\left(\frac{3\alpha_a^{(1)}}{\alpha_a^{(0)}}\right)}q \tag{23}$$

so that at the surface

$$I^{(+)} = \epsilon\cdot 4\pi B(T_w) + (1 - \epsilon)\,I^{(-)}. \tag{24}$$

The result is that which Cess [7] derived without noting that it is a direct consequence of the physical restrictions imposed by the differential approximation. The average direction cosine of the propagation vector, $\sqrt{(\alpha_a^{(0)}/3\alpha_a^{(1)})}$, is weighted by the anisotropy of the situation. Equation (22), which is particularly useful in iterative schemes [8], could not be obtained in this manner.

In the planar case the exact formulation in a quasi-isotropic ($\alpha_a^{(0)} = \alpha_a^{(1)} = \alpha_a$) gas leads to

$$I_{0_w} + \left(\frac{2-\epsilon}{\epsilon}\right) Cq_w = 2\pi\epsilon B(T_w)\, K(\epsilon)$$

$$+ 2(2-\epsilon)\,\tau_l \int_0^\infty \pi \frac{\alpha_e J}{\alpha_a} \{E_1(\tau_l\eta') - CE_2(\tau_l\eta')\}\, d\eta' \tag{25}$$

$$K(\epsilon) = E_2(0) + C\left(\frac{2-\epsilon}{\epsilon}\right) E_3(0)$$

where C is an arbitrary function of time and $E_n(x)$, $n = 1, 2, 3$ are exponential integrals. Equations (21) and (22) are consistent with the substitution $E_2(x) = \exp[-(\sqrt{3})\,x]$ [9] in which case $C = (\sqrt{3})$ will remove the integral from equation (25) if the recursion relations among the $E_n(x)$ are employed. The commonly used Marshak boundary condition is a consequence of the substitution $E_2(x) = e^{-2x}$, $C = 2$, and is consistent only with a forward-reverse (P_0 half-range) approximation [10] which leads to an erroneous optically thick limit. It has been demonstrated [8] that when absorption of shock layer radiation by the upstream gas is included in problems of RGD which are investigated with equation (1) and (2) *only* a Mark-type emission condition in the free stream avoids divergence from radiative equilibrium. Furthermore, only equation (21) yields nearly correct heat fluxes at general surfaces [8, 11]. Thus only the Mark condition can be recommended for use in RGD.

BOUNDARY CONDITIONS FOR GENERAL GEOMETRIES

The principles outlined for one-dimensional fields in the previous section may be applied to the derivation of boundary conditions in any situation. For example, in a two-dimensional field in which the surface of interest has a radius of curvature $R(s)$ the governing equations are

$$\left(\frac{R}{R+n}\right)\left\{\frac{\partial q_s}{\partial s} + \frac{\partial}{\partial n}\left(q_n \cdot \frac{R+n}{R}\right)\right\} = \tau_l\{4\pi\alpha_e J - \alpha_a^{(0)} I_0\} \tag{26a}$$

$$\frac{R}{R+n}\frac{\partial I_0}{\partial s} = -3\tau_l \alpha_a^{(1)} q_s \tag{26b}$$

$$\frac{\partial I_0}{\partial n} = -3\tau_l \alpha_a^{(1)} q_n \tag{26c}$$

where n and s are coordinates normal to and along the surface. If ϕ is defined as before it may be shown that

$$\frac{\partial^2\phi}{\partial\bar{\eta}^2} - \phi = F(s, \bar{\eta}) \tag{27a}$$

where

$$\bar{\eta} = (\sqrt{3})\,\tau_l \int_0^n \sqrt{[\alpha_a^{(0)}(s, n')\,\alpha_a^{(1)}(s, n')]}\, dn' \quad \text{and,}$$

$$F(s, \bar{\eta}) = \frac{4\pi\alpha_e}{3\tau_l \alpha_a^{(0)}} J$$

$$-\frac{1}{3\tau_l^2}\left\{\frac{\partial\phi}{\partial\eta}\left[\frac{\partial}{\partial\eta}\left(\log\sqrt{\left(\frac{\alpha_a^{(0)}}{\alpha_a^{(1)}}\right)}\right) + \frac{1}{(R+n)\sqrt{(\alpha_a^{(0)}\alpha_a^{(1)})}}\right] + \frac{R}{\alpha_a^{(0)}(R+n)}\left[\frac{\partial}{\partial s}(q_s) + \frac{\partial\eta}{\partial s}\frac{\partial}{\partial\eta}(q_s)\right]\right\}. \tag{27b}$$

The division of terms between the left- and right-hand sides of equation (27a) is at our disposal as long as the terms retained on the left are sufficient to allow satisfaction of boundary conditions far from the disturbance and those on the right can be manipulated within the isolated gas with physically reasonable assumptions about its absorption properties. Only by concentrating on the normal direction can one satisfy both requirements. Note that because the absorption coefficients vary with both n and s the tangential flux is given by

$$q_s = \left(\frac{\partial\phi}{\partial s} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial s}\right)\left(\frac{R}{R+n}\right)\frac{1}{\alpha_a^{(1)}}.$$

The flux normal to the surface is still related only to $\partial\phi/\partial\eta$, and both diffusely reflecting and black surfaces may be constructed as before. The one-dimensional Green's function is still appropriate, and we find that

$$I_{0_w} + \left(\frac{2-\epsilon}{\epsilon}\right)\sqrt{\left(\frac{3\alpha_a^{(1)}}{\alpha_a^{(0)}}\right)} q_{n_w} = 4\pi B(T_w). \tag{28}$$

Davison [4] has conjectured that since the interaction of radiation with material boundaries is a local phenomenon, its mathematical statement should be independent of geometry. It is clear from equation (28) that this is so.

OPTICALLY THICK SLIP CONDITIONS

The analogy between radiative transfer in the Rosseland limit and molecular conduction has long been recognized. Probstein *et al.* [12, 13] have sought expressions for the temperature slip in the form common in molecular gas-dynamics

$$T(0) - T_w = \kappa\vec{n} \cdot \text{grad}\,(T) \tag{29}$$

where $T(0)$ is the temperature of the gas at the wall, and the slip coefficient, κ, was determined from matching with an optically thin result before catastrophy occurred in the thick limit. A consistent expression for κ may be derived from the general boundary condition of the form of equation (21).

$$I_{0_w} + \left(\frac{2-\varepsilon}{\varepsilon}\right) C\sqrt{\left(\frac{\alpha_e^{(0)}}{\alpha_a^{(1)}}\right)} q_w = 4T_w^4. \tag{30}$$

If $\tau_l \gg 1$ methods similar to those employed by Cess [1] lead to

$$4\{T_{(0)}^4 - T_w^4\} = C_1\left(\frac{2-\epsilon}{\epsilon}\right)\frac{4}{3\alpha_r\tau_l}\vec{\eta} \cdot \text{grad}\,(T^4) + 0\left(\frac{1}{\tau_l^2}\right) \tag{31}$$

where α_R is the Rosseland mean absorption coefficient. The expression derived by Cess applies only to planar surfaces and relies upon Cheng's application of the Marshak condition. Both Cess' and Deissler's [14] versions of equation (31) apply only if the gas has constant absorption coefficients. Equation (31) is not so restricted.

Since the gas temperature at the wall differs little from T_w in this limit, it follows that

$$T_{(0)} - T_w \cong C\left(\frac{2-\epsilon}{\epsilon}\right)\frac{1}{3\tau_l\alpha_R}\vec{n}\cdot\text{grad}\,T \qquad (32)$$

from which the slip coefficient may be identified.

$$\kappa = \left(\frac{2-\epsilon}{\epsilon}\right)\frac{C_1}{3\alpha_R}. \qquad (33)$$

Since this is exactly the form which applies in kinetic theory, the analogy of emissivity with thermal accommodation coefficients is further confirmed. For a black wall and Marshak's condition, $C_1 = 2$, equation (33) is identically Probstein's result. The agreement of his prediction with that of a P_1-approximation to which Marshak's boundary condition was applied is thus to be expected. Temperature slip is always present in RGD when molecular transport phenomena are ignored; thus no special treatment is necessary if one is consistent in the use of equation (21).

CONCLUSIONS

The Mark boundary condition of neutron transport theory may be extended to non-black surfaces by a method of images. The method is restricted neither to grey, non-scattering gases nor to LTE and may be applied to general geometries. Only the Mark condition is consistent both with the governing equations and with the physical restrictions imposed by the differential approximation. It has been noted that temperature jump conditions in the Rosseland limit are a consistent result of more general interactions of radiation with material boundaries in completely general situations. Finally, a definite relationship between the concepts of emissivity and thermal accommodation coefficient has been noted.

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MORE ON GENERALIZING THE DEFINITIONS OF "HEAT" AND "ENTROPY"

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NOMENCLATURE

c_i, concentration of species i, per unit volume;
$\hat{C}_p$, $\partial\hat{H}/\partial T)_{p,\text{composition}}$;
$\hat{E}$, specific energy;
$\hat{H}$, specific enthalpy;
H_i, partial enthalpy for species i;
J_i, flux of i relative to v;
p, pressure;
q, $\varepsilon - v\cdot\pi$;
Q, net energy addition to system by heat transports;

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